# LIMITS OF ABSTRACT ELEMENTARY CLASSES

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Abstract. We show that the category of abstract elementary classes (AECs) and concrete functors is closed under constructions of "limit type", which generalizes the approach of Mariano, Zambrano and Villaveces away from the syntactically oriented framework of institutions. Moreover, we provide a broader view of this closure phenomenon, considering a variety of categories of accessible categories with additional structure, and relaxing the assumption that the morphisms be concrete functors.

## 1. Introduction

One of the main virtues of accessible categories is that they are closed under constructions of limit type ([9]). This should be made precise by considering accessible functors between accessible categories and showing that the resulting 2-category is closed under appropriate limits. These limits can be reduced to products, inserters and equifiers and are called PIE-limits. Proofs of this result (see [9], or [1]) also show that the category of accessible categories with directed colimits and functors preserving directed colimits is closed under PIE-limits. The needed 2-categorical limits are explained both in [9] and [1] and we recommend [4] for a more systematic introduction.

Recent papers [3], [6] and [7] have shown that abstract elementary classes ([2]) can be understood as special accessible categories with directed colimits. In [7], in particular, the authors develop a hierarchy of such categories, extending from accessible categories with directed colimits to AECs themselves. Here we show that each stage in this hierarchy is closed under PIE-limits as well, provided we take the morphisms to be directed colimit preserving functors. This closure becomes more problematic if we insist that the morphisms be concrete functors: here we see that the iso-fullness axiom for AECs (heretofore unneeded

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in the category-theoretic analysis thereof) is essential to guarantee the existence of desired limits.

Schematically, our results encompass the categories in the figure below, where the downward-accumulating properties of the objects are described in the left margin, and the properties of the morphisms are listed at the top.

$$
\begin{tikzcd}
 & \text{Preserve directed colimits} & \text{Subconcrete} & \text{Concrete} \\
\text{Accessible} & \mathbf{Acc} \arrow[d, dash] & & \\
\text{Directed colimits} & \mathbf{Acc}_0 \arrow[d, dash] & & \\
\text{Concrete directed colimits} & \mathbf{Acc}_1 \arrow[r, dash] \arrow[d, dash] & \mathbf{Acc}_1^\dagger \arrow[r, dash] \arrow[d, dash] & \mathbf{Acc}_1^* \arrow[d, dash] \\
\text{Coherent, concrete monos} & \mathbf{Acc}_2 \arrow[r, dash] \arrow[d, dash] & \mathbf{Acc}_2^\dagger \arrow[r, dash] \arrow[d, dash] & \mathbf{Acc}_2^* \arrow[d, dash] \\
\text{Iso-full, replete} & \mathbf{Acc}_3 \arrow[r, dash] & \mathbf{Acc}_3^\dagger \arrow[r, dash] & \mathbf{Acc}_3^*
\end{tikzcd}
$$

Subconcrete functors, introduced in Definition 3.1 below, are a natural generalization of the concrete case. We show that all the pictured categories are closed under PIE-limits in **Acc**, with the exception of $\mathbf{Acc}_3$, $\mathbf{Acc}_1^*$ and $\mathbf{Acc}_3^*$. The last category has PIE-limits but it is closed in **Acc** only under inserters and equifiers while products are calculated in $\mathbf{Acc} \downarrow \mathbf{Set}$. We note that the objects in categories along the bottom row are (equivalent to) AECs, but equipped with three different notions of morphism, ranging from the most general—functors preserving directed colimits—to a very close generalization of the syntactically-derived functors in [10], namely directed-colimit preserving functors that are concrete, i.e. respect underlying sets. In particular, the closure result corresponding to the bottom right entry is the promised generalization of [10], shifting it out of the framework of institutions and into a more intrinsic, purely syntax-free characterization. We consider the precise relationship between our result and that of [10] in Remark 3.4.

In fact, our ambitions are broader: inspired by the example of metric AECs, in which directed colimits need not be concrete but $\aleph_1$-directed

colimits always are, we consider a second version of this diagram in which we require only that the categories from the third row down have concrete $\kappa$-directed colimits for a given $\kappa$—such categories will be distinguished by the superscript $\kappa$. In particular, the category $\mathbf{Acc}_3^{\dagger\kappa}$ will consist of $\kappa$-CAECs as defined in [8], with subconcrete functors as morphisms. We obtain a closure result there as well.

## 2. Accessible categories with directed colimits

Recall that a category $\mathcal{K}$ is $\lambda$-*accessible*, $\lambda$ a regular cardinal, if it has $\lambda$-directed colimits (i.e. colimits indexed by a $\lambda$-directed poset) and contains, up to isomorphism, a set $\mathcal{A}$ of $\lambda$-presentable objects such that each object of $\mathcal{K}$ is a $\lambda$-directed colimit of objects from $\mathcal{A}$. Here, an object $K$ is $\lambda$-*presentable* if its hom-functor $\mathcal{K}(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves $\lambda$-directed colimits. A category is *accessible* if it is $\lambda$-accessible for some $\lambda$. A functor $F : \mathcal{K} \to \mathcal{L}$ between $\lambda$-accessible categories is called $\lambda$-accessible if it preserves $\lambda$-directed colimits. $F$ is called *accessible* if it is $\lambda$-accessible for some $\lambda$. In this way, we get the category $\mathbf{Acc}$ whose objects are accessible categories and morphisms are accessible functors.

**Remark 2.1.** We work in the Gödel-Bernays set theory. Thus a category $\mathcal{K}$ is a class of objects together with a class $\mathcal{K}(A, B)$ of morphisms $A \to B$ for each object $A$ and $B$. It is called *locally small* if all $\mathcal{K}(A, B)$ are sets. Any accessible category is locally small. It is important to observe that $\mathbf{Acc}$ is a category which is not locally small. The reason is that a $\lambda$-accessible functor $F : \mathcal{K} \to \mathcal{L}$ is determined by its restriction on the full subcategory $\mathcal{A}$ of $\lambda$-presentable objects.

We may regard $\mathbf{Acc}$ as a 2-category where the 2-cells are natural transformations. As noted above, $\mathbf{Acc}$ is closed under appropriate 2-limits, namely PIE-limits, where "PIE" abbreviates "products," "inserters" and "equifiers." This means that these 2-limits exist in $\mathbf{Acc}$ and are calculated in the non-legitimate category $\mathbf{CAT}$ of categories, functors and natural transformations. It follows that $\mathbf{Acc}$ is closed under lax limits and under pseudolimits (see [9] or [1]).

Recall that, given functors $F, G : \mathcal{K} \to \mathcal{L}$, the *inserter category* $\mathrm{Ins}(F, G)$ is the subcategory of the comma category $F \downarrow G$ consisting

of all objects $f : FK \to GK$ and all morphisms

$$\begin{tikzcd} FK \arrow[r, "f"] \arrow[d, "Fk"'] & GK \arrow[d, "Gk"] \\ FK' \arrow[r, "f'"'] & GK' \end{tikzcd}$$

The *projection functor* $P : \mathrm{Ins}(F, G) \to \mathcal{K}$ sends $f : FK \to GK$ to $K$. The universal property of $\mathrm{Ins}(F, G)$ is the existence of a natural transformation $\varphi : FP \to GP$ (given as $\varphi_{Pf} = f$) in the sense that for any $H : \mathcal{H} \to \mathcal{K}$ with $\psi : FH \to GH$ there is a unique $\bar{H} : \mathcal{H} \to \mathrm{Ins}(F, G)$ such that $P\bar{H} = H$ and $\varphi H = \psi$ (see [4]). Since **Acc** is full in **CAT** with respect to 2-cells, we can ignore the 2-dimensional aspect of universality.

Given functors $F, G : \mathcal{K} \to \mathcal{L}$ and natural transformations $\varphi, \psi : F \to G$, the *equifier* $\mathrm{Eq}(\varphi, \psi)$ is the full subcategory of $\mathcal{K}$ consisting of all objects $K$ such that $\varphi_K = \psi_K$. Let $P : \mathrm{Eq}(\varphi, \psi) \to \mathcal{K}$ be the inclusion. The universal property of $\mathrm{Eq}(\varphi, \psi)$ is that $\varphi P = \psi P$ and for any $H : \mathcal{H} \to \mathcal{K}$ with $\varphi H = \psi H$ there is a unique $\bar{H} : \mathcal{H} \to \mathcal{K}$ such that $P\bar{H} = H$ (see [4]); the 2-dimensional aspect of universality can be ignored again.

We now consider accessible categories having all directed colimits. Let $\mathbf{Acc}_0$ be the 2-category whose objects are accessible categories with directed colimits, morphisms are functors preserving directed colimits and 2-cells are natural transformations.

**Theorem 2.2.** $\mathbf{Acc}_0$ *is closed under PIE-limits in* **Acc**.

*Proof.* Let $\mathcal{K}_i$, $i \in I$ be accessible categories with directed colimits. Following [1] 2.67, the product $\prod_{i \in I} \mathcal{K}_i$ is an accessible category. Clearly, it has all directed colimits and the projections $P_i : \prod \mathcal{K}_i \to \mathcal{K}_i$ preserve them. Let $\mathcal{L}$ be an accessible category with directed colimits and $Q_i : \mathcal{L} \to \mathcal{K}_i$ functors preserving directed colimits. Then the induced functor $\mathcal{L} \to \prod \mathcal{K}_i$ preserves directed colimits. Hence $\prod \mathcal{K}_i$ is the product in $\mathbf{Acc}_0$.

Let $\mathcal{K}, \mathcal{L}$ be accessible categories with directed colimits and $F, G : \mathcal{K} \to \mathcal{L}$ be functors preserving directed colimits. Following [1] 2.72, $\mathrm{Ins}(F, G)$ is an accessible category which clearly has directed colimits. Let $\mathcal{H}$ be an accessible category with directed colimits, $H : \mathcal{H} \to \mathcal{K}$ preserve directed colimits and and $\psi : FH \to GH$ a natural transformation. Then the induced functor $\bar{H} : \mathcal{H} \to \mathrm{Ins}(F, G)$ preserves directed colimits. Hence $\mathrm{Ins}(F, G)$ is an inserter in $\mathbf{Acc}_0$.

Finally, let $\mathcal{K}, \mathcal{L}$ be accessible categories with directed colimits, $F, G : \mathcal{K} \to \mathcal{L}$ functors preserving directed colimits and $\varphi, \psi : F \to G$ natural transformations. Following [1] 2.76, $\mathrm{Eq}(\varphi, \psi)$ is an accessible category. Again, it is clear that $\mathrm{Eq}(\varphi, \psi)$ has all directed colimits. Let $\mathcal{H}$ be an accessible category with directed colimits and $H : \mathcal{H} \to \mathcal{K}$ a functor preserving directed colimits with $\varphi H = \psi H$. Then the induced functor $\bar{H} : \mathcal{H} \to \mathrm{Eq}(\varphi, \psi)$ preserves directed colimits. Hence $\mathrm{Eq}(\varphi, \psi)$ is an equifier in $\mathbf{Acc}_0$. □

We say that $(\mathcal{K}, U)$ is an *accessible category with concrete directed colimits* if $\mathcal{K}$ is an accessible category with directed colimits and $U : \mathcal{K} \to \mathbf{Set}$ is a faithful functor to the category of sets that preserves directed colimits. Let $\mathbf{Acc}_1$ be the full sub-2-category of $\mathbf{Acc}_0$ consisting of accessible categories with concrete directed colimits. In particular, morphisms in $\mathbf{Acc}_1$ are functors preserving directed colimits.

**Theorem 2.3.** $\mathbf{Acc}_1$ *is closed under PIE-limits in* $\mathbf{Acc}$.

*Proof.* We must show that PIE-limits of accessible categories with concrete directed colimits have concrete directed colimits. This is evident for inserters and equifiers because, in the first case, the projection functor $P : \mathrm{Ins}(F, G) \to \mathcal{K}$ is faithful and, in the second case, $\mathrm{Eq}(\varphi, \psi)$ is a full subcategory of $\mathcal{K}$. Consider accessible categories with concrete directed colimits $(\mathcal{K}_i, U_i)$, $i \in I$. Then the functor $U : \prod_{i \in I} \mathcal{K}_i \to \mathbf{Set}$ sending $(A_i)_{i \in I}$ to $\coprod_{i \in I} U_i A_i$ is faithful. Since

$$\operatorname{colim} \coprod_{i \in I} U_i A_i \cong \coprod_{i \in I} \operatorname{colim} U_i A_i,$$

$\prod_{i \in I} \mathcal{K}_i$ is an accessible category with concrete directed colimits. □

**Remark 2.4.** (1) We could also consider the subcategory $\mathbf{Acc}_1^*$ having the same objects as $\mathbf{Acc}_1$ but whose morphisms are *concrete* functors $F : \mathcal{K}_1 \to \mathcal{K}_2$ preserving directed colimits. By "concrete," we mean that $F$ commutes with the relevant underlying set functors, i.e. $U_2 F = U_1$. The category $\mathbf{Acc}_1^*$ is closed in $\mathbf{Acc}$ under inserters and equifiers but not under products. In fact, we are in the comma category $\mathbf{Acc} \downarrow \mathbf{Set}$ where $\prod_{i \in I} (\mathcal{K}_i, U_i)$ is the multiple pullbacks of $U_i$ over $\mathbf{Set}$. While $\mathbf{Acc}$ has multiple *pseudopullbacks*, it does not have multiple pullbacks. For multiple pullbacks, we would need all of the functors $U_i$ to be *transportable* in the sense that for any isomorphism $f : U_i A \to X$ there is a unique isomorphism $\overline{f} : A \to B$ such that $U_i(\overline{f}) = f$ (this also implies $U_i \overline{B} = X$). Then a multiple pullback of $U_i$ is equivalent to their multiple pseudopullback and thus it belongs to $\mathbf{Acc}$. This is done for a pullback in [9] 5.1.1 and the multiple case is analogous.

(2) Theorem 2.3 is also valid for the full sub-2-category $\mathbf{Acc}_1^\kappa$ of $\mathbf{Acc}_0$ consisting of accessible categories with directed colimits where $\kappa$-directed colimits are concrete. These categories appear in [8] in connection with metric abstract elementary classes.

An accessible category $(\mathcal{K}, U)$ with concrete directed colimits is *coherent* if for each commutative triangle

$$\begin{tikzcd} UA \arrow[rr, "U(h)"] \arrow[dr, "f"'] & & UC \\ & UB \arrow[ur, "U(g)"'] & \end{tikzcd}$$

there is $\overline{f} : A \to B$ in $\mathcal{K}$ such that $U(\overline{f}) = f$.

We say that morphisms of $\mathcal{K}$ are *concrete monomorphisms* if any morphism of $\mathcal{K}$ is a monomorphism which is preserved by $U$. Let $\mathbf{Acc}_2$ be the full sub-2-category of $\mathbf{Acc}_1$ consisting of coherent accessible categories with concrete monomorphisms.

**Theorem 2.5.** $\mathbf{Acc}_2$ *is closed under PIE-limits in* $\mathbf{Acc}$.

*Proof.* Since there is no problem with concrete monomorphisms, we have to show that PIE-limits of coherent accessible categories are coherent. This is evident for equifiers because $\mathrm{Eq}(\varphi, \psi)$ is a full subcategory of $\mathcal{K}$. Consider coherent accessible categories $(\mathcal{K}_i, U_i)$, $i \in I$. We have to show that $U : \prod_{i\in I} \mathcal{K}_i \to \mathbf{Set}$ sending $(A_i)_{i\in I}$ to $\coprod_{i\in I} U_i A_i$ is coherent. Consider a commutative triangle

$$\begin{tikzcd} U(A_i) \arrow[rr, "U(h)"] \arrow[dr, "f"'] & & U(C_i) \\ & U(B_i) \arrow[ur, "U(g)"'] & \end{tikzcd}$$

and $a \in U_i A_i$. Assume that $fa_i \in U_j B_j$ for $j \neq i$. Then $(Ug)fa_i \in U_j C_j$ and $(Uh)a_i \in U_i C_i$, which is impossible. Thus $f = \coprod_{i\in I} f_i$. Since each $U_i$ is coherent, there are morphisms $\overline{f}_i : A_i \to B_i$ such that $U(\overline{f}_i) = f$. Hence $\prod_{i\in I} \mathcal{K}_i$ is coherent.

Consider morphisms $F, G : \mathcal{K} \to \mathcal{L}$ in $\mathbf{Acc}_2$. We have to show that the composition

$$\mathrm{Ins}(F, G) \xrightarrow{\ P\ } \mathcal{K} \xrightarrow{\ U\ } \mathbf{Set}$$

is coherent. Consider a commutative triangle

$$\begin{tikzcd} UPf_1 \arrow[rr, "UP(h)"] \arrow[dr, "f"'] & & UPf_3 \\ & UPf_2 \arrow[ur, "UP(g)"'] & \end{tikzcd}$$

where $f_i : FK_i \to GK_i$, $i = 1, 2, 3$. Thus we have a commutative triangle

$$\begin{tikzcd} UK_1 \arrow[rr, "U(h)"] \arrow[dr, "f"'] & & UK_3 \\ & UK_2 \arrow[ur, "U(g)"'] & \end{tikzcd}$$

and, since $U$ is coherent, we have $f = U\overline{f}$. Thus we get the diagram

$$\begin{tikzcd} FK_1 \arrow[r, "f_1"] \arrow[d, "F\overline{f}"'] & GK_1 \arrow[d, "G\overline{f}"] \\ FK_2 \arrow[r, "f_2"] \arrow[d, "Fg"'] & GK_2 \arrow[d, "Gg"] \\ FK_3 \arrow[r, "f_3"'] & GK_3 \end{tikzcd}$$

where the outer rectangle and the bottom square commute. Since $Gg$ is a monomorphism, the upper square commutes as well. Hence $\overline{f} : f_1 \to f_2$ is a morphism in $\mathrm{Ins}(F, G)$ and $f = UP\overline{f}$. Therefore $PU$ is coherent. □

**Remark 2.6.** (1) The assumption that objects of $\mathbf{Acc}_2$ have concrete monomorphisms was needed in the proof of closure under inserters.

(2) Theorem 2.5 is also valid for the full sub-2-category $\mathbf{Acc}_2^\kappa$ of $\mathbf{Acc}_1^\kappa$ consisting of coherent accessible categories with directed colimits and concrete monomorphisms.

Abstract elementary classes can be characterized as coherent accessible categories $\mathcal{K}$ with directed colimits and with concrete monomorphisms satisfying two additional conditions dealing with finitary function and relation symbols interpretable in $\mathcal{K}$ (see [7]). Here, *finitary relation symbols interpretable in* $\mathcal{K}$ are subfunctors $R$ of $U^n = \mathbf{Set}(n, U-)$ where $n$ is a finite cardinal. *Finitary function symbols interpretable in*

$\mathcal{K}$ are natural transformations $h : U^n \to U$. Since $n$-ary function symbols can be replaced by $(n+1)$-ary relation symbols, we can confine ourselves to finitary relation symbols interpretable in $\mathcal{K}$. Let $\Sigma_{\mathcal{K}}$ consists of those finitary relation symbols $R$ interpretable in $\mathcal{K}$ for which $\mathcal{K}$-morphisms $f : A \to B$ behave as embeddings. This means that if $(Uf)^n(a) \in R_B$ then $a \in R_A$. We get the functor $E : \mathcal{K} \to \mathbf{Emb}(\Sigma_{\mathcal{K}})$ where $\mathbf{Emb}(\Sigma_{\mathcal{K}})$ is the category of $\Sigma_{\mathcal{K}}$-structures whose morphisms are substructure embeddings. Now, $\mathcal{K}$ is an abstract elementary class if and only if the functor $E$ is *full with respect to isomorphisms* and *replete*. The first condition means that if $f : EA \to EB$ is an isomorphism then there is an isomorphism $\overline{f} : A \to B$ with $E\overline{f} = f$. We also say that $R \in \Sigma_{\mathcal{K}}$ *detect isomorphisms*; this condition makes $\mathcal{K}$ equivalent to an abstract elementary class. The second condition means that if $EA$ is isomorphic to $X$ then there is $B \in \mathcal{K}$ such that $A$ is isomorphic to $B$ and $EB = X$.

We note that abstract elementary classes are commonly presented via an embedding $\mathcal{K} \to \mathbf{Emb}(\Sigma)$. In this case, $\Sigma \subseteq \Sigma_{\mathcal{K}}$ and, in fact, $\Sigma_{\mathcal{K}}$ is the largest relational signature in which $\mathcal{K}$ can be presented.

Let $\mathbf{Acc}_3$ be the full sub-2-category of $\mathbf{Acc}_2$ consisting of categories equivalent to abstract elementary classes.

**Proposition 2.7.** $\mathbf{Acc}_3$ *is closed under products and equifiers in* $\mathbf{Acc}$.

*Proof.* The closedness under equifiers immediately follows from the fact that $\mathrm{Eq}(\varphi, \psi) \to \mathcal{K}$ is a replete, full embedding. Consider $(\mathcal{K}_i, U_i)$, $i \in I$, in $\mathbf{Acc}_3$. Given $n$-ary relation symbols $R_i \in \Sigma_{\mathcal{K}_i}$ where $i \in I$, we get the $n$-ary relation symbol $R = \coprod_i R_i$ belonging to $\Sigma_{\prod_i \mathcal{K}_i}$. It includes unary interpretable relation symbols given by $R_j = U_j$ and $R_i = \emptyset$ for $i \neq j$. It is easy to see that these $R$ detect isomorphisms. Thus $E : \prod_i \mathcal{K}_i \to \mathbf{Emb}(\Sigma_{\prod \mathcal{K}_i})$ is full with respect to isomorphisms. Clearly, it is replete. □

**Remark 2.8.** In the case of inserters, any finitary relation symbol $R$ interpretable in $\mathcal{K}$ yields the finitary relation symbol $RP$ interpretable in $\mathrm{Ins}(F, G)$. Let $f_i : FK_i \to GK_i$ for $i = 1, 2$ and

$$f : UK_1 = UPf_1 \to UPf_2 = UK_2$$

be a bijection such that $f^n$ induces a bijection between $Sf_1$ and $Sf_2$ for each $n$-ary relation symbol $S$ interpretable in $\mathrm{Ins}(F, G)$. By taking $S = RP$ we get that $f^n$ induces a bijection between $RK_1$ and $RK_2$ for each $n$-ary relation symbol $R$ interpretable in $\mathcal{K}$. Since $E : \mathcal{K} \to \mathbf{Emb}(\Sigma_{\mathcal{K}})$ is full with respect to isomorphisms, there is an isomorphism $\overline{f} : K_1 \to K_2$ with $U\overline{f} = f$. But we do not know whether $\overline{f} : f_1 \to f_2$

is a morphism, i.e., whether the square

$$\begin{tikzcd} FK_1 \arrow[r, "f_1"] \arrow[d, "F\overline{f}"'] & GK_1 \arrow[d, "G\overline{f}"] \\ FK_2 \arrow[r, "f_2"'] & GK_2 \end{tikzcd}$$

commutes.

**Problem 2.9.** Is $\mathbf{Acc}_3$ closed under inserters in $\mathbf{Acc}$?

## 3. Abstract elementary classes

**Definition 3.1.** Let $(\mathcal{K}_1, U_1)$ and $(\mathcal{K}_2, U_2)$ be concrete categories. We say that a functor $H : \mathcal{K}_1 \to \mathcal{K}_2$ is *subconcrete* if there is a natural monotransformation $\alpha : U_2H \to U_1$ such that if $(U_1 f)a \in U_2HB$ then $a \in U_2HA$ for each $f : A \to B$ in $\mathcal{K}_1$.

This means that $U_2H$ is a unary relation symbol belonging to $\Sigma_{\mathcal{K}_1}$. Any concrete functor is subconcrete. Since a composition of subconcrete functors is subconcrete, we get the subcategory $\mathbf{Acc}_1^\dagger$ of $\mathbf{Acc}_1$ consisting of accessible categories with concrete directed colimits and subconcrete functors preserving directed colimits. Analogously, we get the full subcategory $\mathbf{Acc}_2^\dagger$ of $\mathbf{Acc}_1^\dagger$ consisting of coherent accessible categories and concrete monomorphisms whose morphisms are subconcrete functors preserving directed colimits. Finally, we have the category

$$\mathbf{Acc}_3^\dagger = \mathbf{Acc}_3 \cap \mathbf{Acc}_2^\dagger$$

of categories equivalent to abstract elementary classes and subconcrete functors preserving directed colimits.

**Theorem 3.2.** $\mathbf{Acc}_1^\dagger$, $\mathbf{Acc}_2^\dagger$ *and* $\mathbf{Acc}_3^\dagger$ *are closed under PIE-limits in* $\mathbf{Acc}$.

*Proof.* In $\mathbf{Acc}_1^\dagger$ and $\mathbf{Acc}_2^\dagger$ the case of equifiers and inserters is evident because $P : \mathrm{Eq}(\varphi, \psi) \to \mathcal{K}$ and $P : \mathrm{Ins}(F, G) \to \mathcal{K}$ are concrete. In the case of products, the projections $P_i : \prod_i \mathcal{K}_i \to \mathcal{K}_i$ are subconcrete – take the coproduct injections $U_iP_i \to U$ where $U : \prod \mathcal{K}_i \to \mathbf{Set}$ is from the proof of 2.3.

We have to prove that $\mathbf{Acc}_3^\dagger$ is closed under inserters. Let $(\mathcal{K}_1, U_1)$ and $(\mathcal{K}_2, U_2)$ be abstract elementary classes and $F, G : \mathcal{K}_1 \to \mathcal{K}_2$ subconcrete functors. First, in the notation of 2.8, we show that the square

$$\begin{tikzcd} FK_1 \arrow[r, "f_1"] \arrow[d, "F\overline{f}"'] & GK_1 \arrow[d, "G\overline{f}"] \\ FK_2 \arrow[r, "f_2"'] & GK_2 \end{tikzcd}$$

commutes. Since $F$ and $G$ are subconcrete, we get unary relation symbols $U_2F, U_2G \in \Sigma_{\mathcal{K}}$. Hence we have unary relation symbols

$$U_2FP, U_2GP \in \Sigma_{\mathrm{Ins}(F,G)}.$$

Thus we have a binary relation symbol $R \in \Sigma_{\mathrm{Ins}(F,G)}$ such that $(a,b) \in R_g$, $g : FK \to GK$, if $a \in U_2FPg$, $b \in U_2GPg$ and $b = (U_2g)a$. To see that the above square commutes, notice that $(a, (U_2f_1)a) \in R_{f_1}$ for each $a \in U_2FK_1$. It follows that $((U_2F\overline{f})a, U_2(G(\overline{f})f_1)a) \in R_{f_2}$, and therefore that $U_2(G(\overline{f})f_1)a = U_2(f_2F\overline{f})a$ for each $a \in U_2FK_1$. Hence

$$U_2(G(\overline{f})f_1) = U_2(f_2F\overline{f})$$

and, since $U_2$ is faithful,

$$G(\overline{f})f_1 = f_2F\overline{f}.$$

It remains to show that $(\mathrm{Ins}(F,G), U_1P)$ is replete. But, having $f : FA \to GA$ in $\mathrm{Ins}(F,G)$ and an isomorphism $h : A \to B$, then $g = G(h)fF(h)^{-1} : FB \to GB$ belongs to $\mathrm{Ins}(F,G)$ and $h : f \to g$ is an isomorphism. □

Finaly, we have the category

$$\mathbf{Acc}_3^* = \mathbf{Acc}_3 \cap \mathbf{Acc}_1^*$$

of categories equivalent to abstract elementary classes whose morphisms are concrete functors preserving directed colimits.

**Theorem 3.3.** *$\mathbf{Acc}_3^*$ has PIE-limits.*

*Proof.* Since concrete functors are subconcrete, $\mathbf{Acc}_3^*$ is closed in $\mathbf{Acc}$ under inserters and equifiers. Products $\prod_{i\in I}(\mathcal{K}_i, U_i)$ are calculated in $\mathbf{Acc} \downarrow \mathbf{Set}$, i.e., they are multiple pullbacks. Since any abstract elementary class $(\mathcal{K}, U)$ has $U$ transportable, multiple pullbacks are equivalent to multiple pseudopullbacks (see 2.4). Following 3.2, $\mathbf{Acc}_3^\dagger$ is closed in $\mathbf{Acc}$ under PIE-limits and, consequently, under pseudolimits. Thus $\prod_{i\in i}(\mathcal{K}_i, U_i)$ belongs to $\mathbf{Acc}_3^*$ and is the product of $(\mathcal{K}_i, U_i)$ there. □

**Remark 3.4.** (1) Let $H : \mathcal{K}_1 \to \mathcal{K}_2$ be a morphism in $\mathbf{Acc}_3^\dagger$. Since $(U_2H)^n$ is an $n$-ary relation symbol belonging to $\Sigma_{\mathcal{K}_1}$, we get an embedding of signatures $\overline{H} : \Sigma_{\mathcal{K}_2} \to \Sigma_{\mathcal{K}_1}$ sending $R$ to $RH$. In particular, it sends $U_2$ to $U_2H$. This induces the subconcrete functor $\mathbf{Emb}(\overline{H}) : \mathbf{Emb}(\Sigma_{\mathcal{K}_1}) \to \mathbf{Emb}(\Sigma_{\mathcal{K}_2})$ given by taking reducts. The square

$$\begin{tikzcd}
\mathcal{K}_1 \arrow[r, "H"] \arrow[d, "E_1"'] & \mathcal{K}_2 \arrow[d, "E_2"] \\
\mathbf{Emb}(\Sigma_{\mathcal{K}_1}) \arrow[r, "\mathbf{Emb}(\overline{H})"'] & \mathbf{Emb}(\Sigma_{\mathcal{K}_2})
\end{tikzcd}$$

clearly commutes.

If $H$ is concrete then $\mathbf{Emb}(\overline{H})$ is concrete as well. This relates our morphisms of abstract elementary classes to the syntactically-derived morphisms considered in [10].

(2) On the other hand, let $G : \Sigma_2 \to \Sigma_1$ be an embedding of signatures. Let $\mathcal{K}_1 \to \mathbf{Emb}(\Sigma_1)$ and $\mathcal{K}_2 \to \mathbf{Emb}(\Sigma_2)$ be abstract elementary classes—in the classical sense—presented in signatures $\Sigma_1$ and $\Sigma_2$: note that, when paired with their natural underlying set functors $U_i : \mathcal{K}_i \to \mathbf{Set}$, they satisfy the purely category-theoretic characterization of AECs following Remark 2.6. Moreover, let $H : \mathcal{K}_1 \to \mathcal{K}_2$ be a functor such that the square

$$\begin{tikzcd}
\mathcal{K}_1 \arrow[r, "H"] \arrow[d] & \mathcal{K}_2 \arrow[d] \\
\mathbf{Emb}(\Sigma_1) \arrow[r, "\mathbf{Emb}(G)"'] & \mathbf{Emb}(\Sigma_2)
\end{tikzcd}$$

commutes. Since $\mathbf{Emb}(G)$ is concrete, $H$ is a morphism in $\mathbf{Acc}_3^*$. These are precisely the morphisms of abstract elementary classes considered in [10].

More generally, consider relational signatures $\Sigma_1, \Sigma_2$ and let $L(\Sigma_1), L(\Sigma_2)$ be the corresponding languages, i.e., sets of all formulas of $\Sigma_1, \Sigma_2$. Consider a mapping $-^* : \Sigma_2 \to \Sigma_1$ of signatures preserving the arity of symbols and let $P$ be a unary relation symbol in $\Sigma_1$. Let $G : L(\Sigma_2) \to \text{Ł}(\Sigma_1)$ be a morphism of languages sending each ($n$-ary) relation symbol $R$ in $\Sigma_1$ to $P^n \wedge R^*$. This defines $G$ on the atomic formulas of $L(\Sigma_1)$; we extend it recursively to all of $L(\Sigma_1)$. In particular, $G$ sends $=$ to the

equality $=_P$ on $P$. Then $\mathbf{Emb}(G) : \mathbf{Emb}(\Sigma_1) \to \mathbf{Emb}(\Sigma_2)$ is a subconcrete functor. Let $\mathcal{K}_1 \to \mathbf{Emb}(\Sigma_1)$ and $\mathcal{K}_2 \to \mathbf{Emb}(\Sigma_2)$ be abstract elementary classes and $H : \mathcal{K}_1 \to \mathcal{K}_2$ be a functor such that the square

$$\begin{tikzcd} \mathcal{K}_1 \arrow[r, "H"] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathbf{Emb}(\Sigma_1) \arrow[r, "\mathbf{Emb}(G)"'] & \mathbf{Emb}(\Sigma_2) \end{tikzcd}$$

commutes. Then $H$ is a morphism in $\mathbf{Acc}_3^\dagger$.

(3) Let $\mathcal{K}$ be the category of infinite sets and monomorphisms. Then $\mathcal{K}$ is an abstract elementary class in the empty signature $\Sigma_2$. Let $\Sigma_1$ contain just $P$ and $=_P$ from (2) and $G : \Sigma_2 \to \Sigma_1$ be the corresponding morphism of languages. We also have $H : \Sigma_1 \to \Sigma_2$ sending $=_P$ to $=$ and $P$ to the formula $x = x$. In this way, we make $\mathcal{K}$ isomorphic to an abstract elementary class in the signature $\Sigma_1$, where we interpret objects of $\mathcal{K}$ as $\Sigma_1$-structures $K$ with $P_K$ infinite and $(\neg P)_K$ countable (see [3], 5.8(3) motivated by [5] 2.10).

(4) Theorem 3.2 is also valid for categories $\mathbf{Acc}_1^{\dagger\kappa}$, $\mathbf{Acc}_2^{\dagger\kappa}$ and $\mathbf{Acc}_3^{\dagger\kappa}$ where $\mathbf{Acc}_1$ is replaced by $\mathbf{Acc}_1^\kappa$.

Analogously, Theorem 3.3 is valid for $\mathbf{Acc}_3^{*\kappa}$.

**Lemma 3.5.** *Any morphism in $\mathbf{Acc}_3^\dagger$ is coherent and transportable.*

*Proof.* Consider the square from 3.4(1). Since the functor $\mathbf{Emb}(\overline{H})$ is coherent, the composition $\mathbf{Emb}(\overline{H})E_1$ is coherent as well. Since $E_2$ is faithful, $H$ is coherent.

Since $\Sigma_{\mathcal{K}_1}$ and $\Sigma_{\mathcal{K}_2}$ contain only relation symbols, the functor $\mathbf{Emb}(\overline{H})$ is surjective on objects and full (by interpreting the missing relations as empty). Consider an isomorphism $f : HA \to B$. We get the isomorphism

$$E_2 f : \mathbf{Emb}(\overline{H})E_1 A = E_2 HA \to E_2 B = \mathbf{Emb}(\overline{H})\tilde{B}.$$

and thus the isomorphism $\tilde{f} : E_1 A \to \tilde{B}$ such that $\mathbf{Emb}(\overline{H})\tilde{f} = E_2 f$. Since $E_1$ is transportable, there is an isomorphism $\overline{f} : A \to \overline{B}$ such that $E_1\overline{B} = \tilde{B}$ and $E_1\overline{f} = \tilde{f}$. Clearly, $H\overline{f} = f$. Thus $H$ is transportable. □

**Remark 3.6.** Following 3.5 and 2.4, pullbacks in $\mathbf{Acc}_3^\dagger$ are equivalent to pseudopullbacks. Thus 3.2 implies that $\mathbf{Acc}_3^\dagger$ is closed in $\mathbf{CAT}$ under pullback. Consequently, the same holds for $\mathbf{Acc}_3^*$, which was proved in [10].

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